Portfolio Selection with Bayesian Risk in Brazilian Stock Market

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Classical Mean-Variance (MV) model with historical covariance has been criticized and it is observed that the choice of covariance matrix has sensitivity to broad parameter estimation. Thus, the present study set out to formulate a new matrix model, using Bayesian inference as a means of estimation in order to replace historical covariance in MV models, called BMM – Bayesian Matrix Model. Our results show that returns earned from May 2002 to December 2009 were superior with BMM in comparison to the MV model with historical covariance and the Ibovespa index as benchmark.

1. Introduction

Economic agents apply resources with a view to obtaining a given return. However, there is a variety of investment alternatives available in the capital market. In choosing between them, diversification plays an important role for investors, because the purpose is to reduce the risk for a given return. Yet, an analysis only from the perspective of return makes this an incomplete assessment, which reflects the need to consider other factors and conditions imposed by the market.

MPT - Modern Portfolio Theory has applications in several market sectors and has been evolving for more than sixty years. Despite the variety of methods and models disseminated, Michaud (1989), Black and Litterman (1992) and Elton and Gruber (1997) have shown that there are gaps that the theory has not been able to solve.

The results of evolution have obtained improvements by increasing the capacity and computational methods used in selecting portfolios, including the increased use of weighting portfolio methods, especially in developed countries, as shown by Battacharya and Galpin (2011). However, some aspects still need to be discussed. Michaud (1989) and Black and Litterman (1992) stated that one of the major factors is the estimation of parameters in the model inputs. Some critical points are observed, mainly in the use of the Mean Variance (MV) model of Markowitz (1952).

According to Michaud (1989) and Black and Litterman (1992), classic MV contains criticism regarding its parameterization. It is observed that the use of variance is not suitable for portfolios that have asymmetric returns. In other words, it does not consider the heavy tails of returns. Another problem is the use of returns, variances and a covariance constant.

According to Bamberg and Spremann (1981) and Fama and French (1989), the volatility of the assets changes constantly, which results in loss of efficiency of the solution proposed by the model over time. This failure causes, according to Bamberg and Spremann (1981) and Michaud (1989), estimation errors, and therefore, changes in the solutions.

The problem of parameter estimation is a factor common to other models developed by the theory. According to Michaud (1989) the coefficients of return and risk have sensitivity to their preparation in relation to the time window selected. This issue relates to the uncertainty of events, which comes from the randomness of asset returns and the randomness of economic, political and social factors.

One solution proposed by the theory is the use of Bayesian inference, in order to model and minimize the expected loss of the portfolio by including decision rules that incorporate other sources of information, not just historical ones. The Bayesian inference, in the selection of portfolios, was used by Mao and Sarndal (1966) as a way to transform a distribution of returns in the past, a priori, into a predictive distribution of returns by using sources of information other than the range of prices. Other models and adaptations that have used the Baysesian inference are Kalymon (1971), Winkler and Barry (1975), Jorion (1986), Frost and Savarino (1986), Shanken (1987), Dumas and Jacquilat (1990), Polson and Tew (2000), Pástor (2000), Bade et al (2009) Greyserman et al (2006) and Tu and Zhou (2010).

In the Brazilian market, Ferreira et al (2009) use the Bayesian inference and adapted the portfolio selection models. Like Mao and Sarndal (1966), these studies use the value of expected loss as a decision criterion for maximizing the utility of the investor. Avramov and Zhou (2010) show, through theoretical evolution, that the performance of Bayesian models for the field has been significant.

Just as Michaud (1989) argues that changes in input parameters lead to different portfolios, so some models propose changes and disturbances in the covariance matrix as shown in the results of Disatnik and Benninga (2007) with the model for a Shrinkage Procedure that causes disturbances in the matrix. Their article, using the example of these innovations and the ability of Bayesian models, set out to formulate a new matrix model for the Bayesian inference, to replace the covariance, using the same structure as the model of Markowitz (1952).

1. BMM – Bayesian matrix model for portfolio selection

Agents use two sources of information, which comes from the randomness of asset prices and that of economic, political and social variables. The models of building portfolios based on Bayesian inference can, according to Avramov and Zhou (2010), incorporate these two sources of information, unlike traditional models that only use information on the assets.

The group of Bayesian models for portfolio selection, according to Avramov and Zhou (2010) use decision theory tools as a way to minimize the error of estimation of parameters. Based on this assumption, Mao and Sarndal (1966) constructed the classical model using the Bayesian inference. One of the great advantages of decision theory is that it includes consideration of the economic variables that can direct the results of the estimates of risk and return.

The model proposed for this research was based on a new matrix to replace the covariance matrix. The new model was based on Bayes' theorem using the assumptions of decision theory in Friedman and Savage (1948) and Berger (1985); the structure of Markowitz (1952); the utility of the return on assets by Mao and Sarndal (1966); the estimator used by Polson and Tew (2000); and the form of preparation of states of nature in Ferreira et al (2009). This study used a combination of ideas in these studies to produce the new model that is now put forward.

According to Berger (1985), the combination of an alternative selection ε and the occurrence of a given state θ causes a particular result c contained in a set of possible consequences $\mathcal{C}$. The relationship $\mathrm{E} \times \Theta \rightarrow \mathcal{C}$ represents the hypothesis that if an action $\varepsilon_i \in \mathrm{E}$ is chosen, then one, and only one, uncertain event will occur $\theta_j \in \Theta$, with a corresponding result $c_{ij} = c(\varepsilon_i, \theta_j) \in \mathcal{C}$.

The uncertainty of the problem associated with the state of nature will affect the alternative of choice. The uncertainties about the occurrence of events or states of nature are measured in probabilities, but these probabilities must be estimated at the time of decision. Bernardo and Smith (1994) show that an alternative is an estimate of the probabilities when the Bayes theorem is used.

In order to apply the theorem, the following elements are needed: (1) the historical frequencies of states of nature $\pi(\Theta)$ given by the probability of each state occurring $p(\theta)$, (2) the frequency of past actions $\pi(E)$ given by the probability of each choice $p(\varepsilon)$ and, (3) the historical frequency of joint occurrences among the choices and states of nature $p(\varepsilon, \theta)$, also called the likelihood.

The likelihood function is a mechanism that aims to combine the historical frequency of the joint occurrence of a state of nature and a chosen alternative. This function is the communication channel between the estimator ε and the state of nature θ. As stated by Berger (1985) and Ferreira et al (2009), the result of this function is a probability matrix with dimension $k * n$, where k is the number of alternatives and n is the number of scenarios of states of nature.

In order to apply the model it is necessary to use the probabilities expected in the time decision, and the distribution expected, or the posterior probabilities and distribution are a result of the application of Bayes' theorem, see Berger (1985) and Scherer (2005). The purpose of this function is to estimate the expected probability $p(\theta|\varepsilon)$ of the states of nature θ as a function of a particular choice of the estimator selected ε, as shown in Equation 01. The expected probability of the states of nature is a component for the preparation of risk and returns, as an element for evaluating the utility of the assets.

$$p(\theta|\varepsilon) = \frac{p(\varepsilon, \theta)p(\theta)}{p(\varepsilon)} \tag{01}$$

The BMM is structured by three basic factors: (1) the returns (r) on assets are the elements observed in the experiment, (2) the states of nature of these assets (θ), and (3) the alternatives of choice based on the estimator (ε). Through this group of elements, the risk is measured in the form of a new matrix for inclusion in the model for structuring the selection of an MV portfolio. The interaction between these elements is shown in Figure 01.

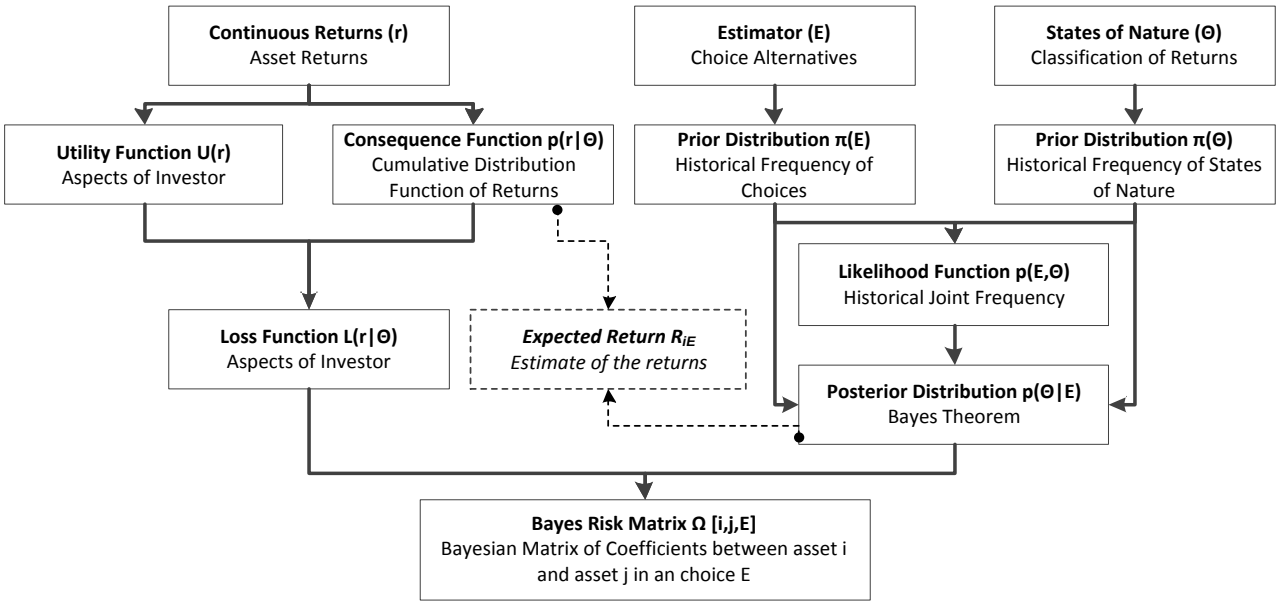

Figure 01 – Framework of BMM – Bayesian Matrix Model

The BMM is initiated by preparing the logarithm of returns on assets, as shown in Equation 02. The cumulative returns can be used in Equation 03. It is worth noting that prices should be adjusted for corporate events, such as dividends, interest, splits and inplits.

$$r_i = \ln(1 + R_i) = \ln\left(\frac{y_n}{y_{n-1}}\right) = (\ln y_n - \ln y_{n-1}) \tag{02}$$

$$r_i[k] = \ln R_i[k] = \ln\left[\prod_{j=1}^{k}(1 + R_i)\right] \tag{03}$$

The first elements in decision theory are alternative actions ε, consisting, according to Berger (1985), of the alternatives that the agent chooses between in order to make a decision. For every decision, there is a set of N alternatives to choose between. In this model, these alternatives are based on a variable called an Estimator that represents the alternative that has the ability to estimate the parameters of the model. The estimator consists of an element of choice that the analyst contributes to predicting the states of nature, and consequently the parameters of risk and return on portfolios.

The Bayesian estimator in this model can be used as a market index as used in Polson and Tew (2000), or also economic series to predict the returns as in Tu and Zhou (2010). The estimator serves as a way to estimate the behavior of the return series in the future, transforming a past distribution, *prior*, into an expected distribution, *posterior*.

For the division of the alternatives or choices, as used by Ferreira et al (2009), observations of the estimator were classified as binary combinations, where the variations of the values of the estimator in the past are transformed into a binary vector, with 1 for positive change and 0 for negative change. The classification of the estimators can contain k scenarios observed, $\mathrm{E_i} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_k\}$. For this model, by using a market index, the alternatives were divided into two, if the market underwent a positive change ε_1 or negative change ε_2, and belonged to the set $\mathrm{E} = \{\varepsilon_1, \varepsilon_2\}$, and $\varepsilon_1 = \{1\}$, $\varepsilon_2 = \{0\}$.

As to the distribution of the last classification, prior distribution is established directly from the relative frequency of the alternatives available in the historical past or through knowledge of market analysts, according to the protocol established by Lins and Souza (2001). The frequency of the estimators is defined by $\pi(\mathrm{E})$, and the sum of the probabilities of each scenario is given by $\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{p}(\varepsilon_\mathrm{i}) = 1$.

In decision theory, the choice of alternatives is complex because it contains uncertain elements associated with the decision, which come mainly from the future state of the observations of the experiment. The second element of decision theory is the set of states of the nature of the observations, according to Bernardo and Smith (1994), which consist of effects or events from the process of decision making. The theta Θ denotes a set of possible states of nature and the number of scenarios is not always known.

The states of nature θ, in the proposed model, come from the matrix of the returns $\mathrm{r_i}$ on each asset i accumulated in $[\mathrm{k}]$ periods, determined by the agent. Since future returns are unknown, this uncertainty prompts the development of a decision model based on historical behavior. The states of Nature of assets are divided into categories as used by Ferreira et al (2009), where the classification is defined as a set of n states, $\Theta_\mathrm{i} = \{\theta_1, \theta_2, \theta_3, \dots, \theta_\mathrm{n}\}$.

The criterion used for constructing states of nature in this model was the transformation of the returns in a binary combination, in which the first three assets of data take a vector of values: these values were obtained in the same way from the estimator. With a series of three asset returns, we obtain nine possible binary combinations for the states, also called scenarios: $\theta_1 = \{0,0,0\}$, $\theta_2 = \{0,0,1\}$, $\theta_3 = \{0,1,0\}$, $\theta_4 = \{0,1,1\}$, $\theta_5 = \{1,0,0\}$, $\theta_6 = \{1,0,0\}$, $\theta_7 = \{1,0,1\}$, $\theta_8 = \{1,1,0\}$ and $\theta_9 = \{1,1,1\}$.

The distribution of the past classification, of the states of nature is estimated directly from their historical frequency or from the knowledge of market analysts, according to the protocol established by Lins and Souza (2001). The frequency

of the historical states of nature is defined by $\pi(\Theta)$, and the sum of the probabilities of each scenario is given by $\sum_{i=1}^{n} p(\theta_i) = 1$.

Besides the estimates of probability, the utility function in the model is used to measure the value attributed by an agent to the series of cumulative returns, in *a* given time window. This function in the model can take different formats as a consequence of the different investor profiles. The utility functions in BMM measure more than just the value of returns, they provide a coefficient for the behavior of the investor before each individual asset is chosen. As an example, see the utility function in Equation 04. The utility of the returns $U(r_{\theta i})$ on asset i given a particular state of nature θ is expressed by the ratio between the maximum and minimum return of assets around its series. (04)

$$U(r_{i\theta}) = \frac{(r_{i\theta} - r_{MIN})}{(r_{MAX} - r_{MIN})}$$

Another element to quantify the value of the utility of the asset is a Consequence Function which is a mechanism to estimate the probability of a given return r_i according to a given choice and a given state of nature. The probability is estimated by using historical returns. It has a mean $\mu_{\theta i}$ and a standard deviation $\sigma_{\theta i}$ in the state of nature θ specified, and it is assumed that the number of returns assumes a normal density, as per Equation 05. The cumulative density function is flexible and can be adapted to other types of distribution of the series of asset returns.

$$p(r_i|\theta) = \text{Normal}(r_i, \mu_{\theta i}, \sigma_{\theta i}) \tag{05}$$

It is possible to combine the Consequence Function and Utility Function so as to design a Loss Function for each individual asset. The loss function or expected loss, according to Berger (1985), consists of measuring the loss in different random situations. Berger (1985) also shows that minimizing the expected loss is equivalent to maximizing the utility of wealth. In the model, the expected loss for each asset is evaluated individually, according to an estimator of the expected scenario and a given state of nature.

Using Equation 06, we can estimate the expected loss by integrating the investor's utility function and cumulative density function, approximately normal, represented by the result of the function presented. The coefficient found is equivalent to the utility of the return. According to Boyle and Tian (2007), to obtain higher returns implies taking higher risks. The estimated coefficients are used as coefficients to be minimized. The Loss Function is equivalent to reducing the expected utility of an asset in relation to the behavior of the distribution of the returns.

$$L_i(r_i, \theta) = \int_{r_{min}}^{r_{max}} U(r_{\theta i}) \cdot p(r_i|\theta) dr_i \tag{06}$$

Based on the concept that risk is the expected loss ratio, the risk of an asset due to a given scenario estimated ε is obtained by using Equation 07. The proposal is to replace the covariance matrix for the model of Markowitz (1952), being represented by the expected loss of each asset in each scenario of the estimator, resulting in $\Omega_{ij\varepsilon}$.

This type of decision rule, as shown in Equation 07, incorporates the posterior distribution of the states of nature on assets $p(\theta|\varepsilon)$ and the result of the expected utility, represented by the Loss Function. This function is a result of the sum of the averages between the losses of assets, considering the state of nature. In other words, the intensity of the loss of assets in the future is estimated by the estimator ε.

The result is a positive semi-definite matrix. Unlike traditional covariance, all values are values greater than the order 10^4. In other words, this matrix minimizes some weak points of the traditional matrix as cited by Bamberg and Spremann (1981) and Michaud (1989).

$$\Omega_{ij\varepsilon} = \sum_{\theta} \left[\frac{L(r_i,\theta)+L(r_j,\theta)}{2}\right] \cdot p(\theta|\varepsilon) \tag{07}$$

The BMM also estimates the expected returns. The proposal is based and founded on the balance of scenarios. The expected Bayesian return, deemed $E(R)_{i\varepsilon}$, is based on the weighted averages of the returns on assets in each state of nature $\mu_{\theta i}$ multiplied by the respective probabilities of these states of nature according to the estimator scenario selected when preparing the risk $p(\theta|\varepsilon)_i$, represented in Equation 08.

$$E(R)_{i\varepsilon} = \sum_{\theta} \mu_{\theta i} \cdot p(\theta|\varepsilon)_i \tag{08}$$

The structure of the BMM was drawn from the structure of the MV model presented in Equations 09, 10, 11 and 12. The equations of the classical model have been replaced by the proposed structure. The risk of the portfolio Ω_p is given by the weighting matrix $\Omega_{ij\varepsilon}$ for each percentage x_i.

Min:

$$\Omega_p = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \Omega_{ij\varepsilon} \tag{09}$$

Subject to.:

$$\sum_{i=1}^{n} E(R)_{i\varepsilon} x_i \geq R_e \tag{10}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{11}$$

$$^{7} \tag{12}$$

$$x_i \geq 0, \qquad x \text{ e } i \in N$$

The constraints of the model are initiated by Equation 10. The return on the portfolio must be greater than the minimum return required by investors. In the first term, the sum of $\sum_{i=1}^{n} E(R)_{i\varepsilon} x_i$ represents the expected return upon sale of the portfolio at the end of the period, in which the weight x_i is multiplied by the returns $E(R)_{i\varepsilon}$. The expected return is prepared by weighting the scenarios, as per Equation 08. Factor R_e is the minimum return required by investors.

Equation 11 is the budget constraint, where the value of the sum must be equal to the amount available to the investor. The fourth constraint in Equation 12 is the non-negativity constraint, without short selling. In this formulation it is the same as the one put forward Markowitz (1952) which considers an investor who at time t, decides what investment portfolio to invest in the time horizon Δt. At time $t + \Delta t$, the investor reconsiders the situation and makes a new decision.

2. Evaluation method

The type of empirical research was of the *ex post facto* format, in a quantitative form as given by Mahoney and Goertz (2006). The evaluation of the results was performed using a sensitivity analysis of the variables of the model. The model was programmed using *Matlab*® software.

The principle method of the evaluation was to check the behavior of the parameters and then the results, return and risk. The analysis was performed by BMM optimizations. As a benchmark we used the traditional MV model and the index of the Brazilian market, IBOVESPA. The sample selected to support evaluating the model included the historical daily closing prices adjusted by dividends, bonuses, splits and inplits. The assets selected were part of the theoretical portfolio of the Bovespa index, valid for the quarters of the years 2002 to 2009. The data were obtained using *Economática*® software.

The effective return on the portfolio was calculated after it had been formed. The return on the portfolio RC_{D+t} at time t, was calculated by the weights x_i and the real returns R_{iD+t} of assets at time t, during the quarter subsequent to the portfolio being formed, as per Equation 13. The cumulative returns were calculated using the product between the returns of period k in Equation 14. The risk of the portfolio during this period was calculated from the standard deviation, as per Equation 15.

$$RC_{D+t} = \sum_{i=1}^{n} R_{iD+t} x_i \tag{13}$$

$$RC[k] = \prod_{j=1}^{k} [1 + RC_{D+j}] \tag{14}$$

(15)

$$\mathrm{Std(RC)} = \sigma_p = \left[\frac{1}{n-1}\sum_{j=1}^{n}\left(\mathrm{RC}_{D+j} - \overline{\mathrm{RC}}\right)^2\right]^{1/2}$$

Besides the main indicators presented, other statistical parameters were calculated such as skewness, kurtosis, median, minimum, maximum, average returns and historical Value at Risk in the period, to a 99% level of significance. The mathematical formulation is described in Equation 16 (Alexander and Baptista (2003)).

$$\mathrm{VaR}_\alpha = \inf\{l \in \Re : p(L > l) \leq 1 - \alpha\} \tag{16}$$

In this study, the analysis of the market index was considered as a benchmark portfolio, the coefficient of Sharpe (1966) was used for comparison, described in Equation 17. In Equation 17, R_p is the portfolio return, and σ_p the standard deviation of returns.

$$\text{Sharpe Index} = \frac{R_p}{\sigma_p} \tag{17}$$

In cases where the indices described showed no perceptible difference, it was necessary to use statistical tests to compare the series. Since it was not possible to prove that the data have a normal distribution, the authors drew on Morgan (2004) who recommends the use of nonparametric tests, because the t test loses its effectiveness. Given that the distribution of the asset returns have heavier tails, it was decided to follow Ribeiro and Leal (2002) who suggest the use of the Wilcoxon test to analyze the significance of the difference between the results.

3. Analysis of results

The BMM was systematized and founded on Decision Theory as set out by Berger (1985) and on utility theory as set out by Friedman and Savage (1948). The modeling was systematized with reference to research studies such as those by Polson and Tew (2000) and Ferreira et al (2009). The first factor to note is the estimator of the model, which is the prediction mechanism of the parameters.

The data used to test the model were the assets that comprise the Bovespa index, and to complement this, the estimator used was the indexer. Although Tu and Zhou (2010) have obtained good results from estimators that use economic variables, this article preferred the market index to simplify the construction procedure, based on the performance obtained by Polson and Tew (2000), who used the S&P500, as the estimator.

In implementing the BMM, the analyst rather than trying to estimate the return and variance of various assets, simply selects the alternative with respect to his market expectations, i.e., he analyzes the market trend. In the case of using the Bovespa index, the market tends to positive change ε_1 or negative ε_2.

Based on the estimator, and the procedures for developing the model described, a positive semi-definite matrix for applying it is developed. The matrix is constructed not by the variability of asset returns, but the probability of the expected loss of assets, individually and collectively. Importantly, the utility function model was applied in the linear, since, according to Berger (1985), this function contributes to the relationship $U(\sum_i r_i) = \sum_i U(r_i)$ of a variable as returns r_i, by eliminating bias in the loss function $L_i(r_i, \theta)$. This function is used as Equation 04.

The first analysis in Figure 2 shows two surface graphs with the coefficients of both matrices, which represent the covariance matrix and the new matrix model, using forty assets. The first is the covariance with misshapen values, with an atypical central point, which shows the extreme points of the correlation coefficient, as discussed in Michaud (1989) and Black and Litterman (1992). This failure leads to a stronger weight in a single asset.

In the second graph in Figure 2, the new matrix presents more atypical points, which shows, in principle, a different standard, but not necessarily a better one. The advantage is that only positive weights are used in a range between 0.9 and 1.2, which shows the elimination of values less than 10^4, and so, mathematically, this reduces the computational effort.

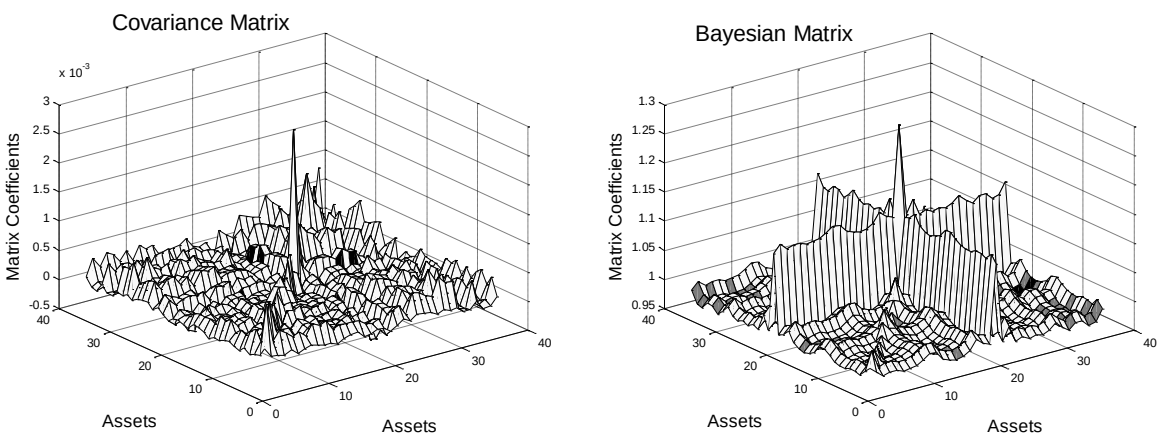

Figure 02 – Coefficients of Covariance and Bayesian Matrices

To evaluate the performance of the model in terms of return and risk the studies by Polson and Tew (2000) and Clarke et al (2011) were used as references. The objective was to analyze the performance of portfolios with the market indicator and the classical model MV model, and tries to improve them.

For evaluation, a historical period of four months, equivalent to an analysis of daily returns in four months, was used which corresponds to the configuration used in the Bovespa Index. The data used were drawn as from the period from 2002 to 2009, but the results obtained for performing the comparison used the period from May 1, 2002 to December 31,

2009. So that the return could be compared using a reliable market indicator, transaction costs were not included, since the market is an indicator of gross score.

The analysis was prepared simultaneously with long-term investment portfolios in the same period as the Bovespa index, i.e. quarterly, using both the BMM and the classic MV. Table 01 presents the descriptive statistics of the actual performance of the portfolios produced by each model, compared with the market index. The required return for the construction of the portfolios was 0.00%, ie the portfolios of both models were executed using the minimum variance, in line with Jobson and Korkie (1980) and Clarke et al (2006, 2011) who suggest doing so, especially because it does not depend on asset returns.

As shown in Table 01, the cumulative return of 510.10% and the average daily return of 0.10% was well above the BMM Ibovespa and the MV model. In the analysis of the risk represented by the standard deviation and the VaR, the classic MV took the lower volatility, thus the BMM obtained an approximate probability of loss. Considering the risk and return together, as per Equation 17, the BMM had the best indicator of the return/risk relationship. These results show that the BMM outperforms the market index in both return and risk, but the classic MV continues to achieve lower risk.

Despite the BMM not showing lower risk, the amplitude of returns had the lowest negative return, as well the smallest positive return, which represents together with other aspects a more elongated distribution than the other series, in leptokurtic format.

The results were presented by the coefficients of skewness, kurtosis, mean and median in addition to which the Kolmogorov-Smirnov test of normality showed that the returns do not have normal distribution, which confirms the proposal of using non-parametric tests to make a comparative analysis of the series.

Table 01 – Descriptive statistics of the effective returns of portfolios built by each model. (1) BMM – Bayesian Matrix Model, (2) MV – Mean Variance Model, and (3) Ibovespa - Bovespa Index.

Statistics of Returns of Portfolios: May 2002 to December 2009			
Coefficients	(1) BMM	(2) MV	(3) Ibovespa
Accumulated Returns (%)	510.10%	239.66%	254.18%
ADR - Average Daily Return (%)	0.10%	0.07%	0.08%
STD - Standard Deviation (%)	1.68%	1.45%	1.98%
VaR 99% sig.	2.66%	2.31%	3.17%
ADR / STD	0.062105	0.049445	0.041865
Asymmetry	0.090775	- 0.310765	- 0.100387
Curtose	2.868287	6.289255	4.666460
Median	0.000137	0.000573	0.000782
Minimum Return (%)	-7.73%	-8.36%	-12.10%
Maximum Return (%)	9.97%	11.20%	13.68%
Average nº of assets in portfolios	5	14	55

The last indicator examined was the average number of assets that were part of the portfolios: the BMM had fewer assets than the MV and the market index. The MV model was consistent with the findings of Thomé Neto (2011) which state that the average is twelve in the Brazilian market. The BMM had portfolios with fewer assets and consequently, greater objectivity in the application of the results, which helps to reduce some types of transaction costs and maintenance (Fabozzi et al (2006)).

Figure 03 shows the cumulative daily return of the Bovespa, the Matrix Model and Bayesian model MV. The period is from May 2002 to 2009. It can be seen that the behavior of the return series is very similar until January 2006, but from then on, the series showed different behaviors.

In the case of the MV model, the period of high returns over the Bovespa index covers the periods between 2006 and 2008 which was a period of economic stability and, in the case of BMM the upside over the Ibovespa was between 2008 and 2009, where there was an economic recession. While models seek to reduce risk, both, together with the market, felt an impact from the subprime crisis, starting in 2008.

Another important aspect in Figure 03, is that the BMM also absorbed less impact in relation to the crisis. The impact of this happened much later. The MV model and the market index had a decrease on their returns in early 2008 and the BMM suffered an impact from it only from the second semester of 2008. It can be concluded that in this case, the BMM showed lower sensitivity to economic dislocation.

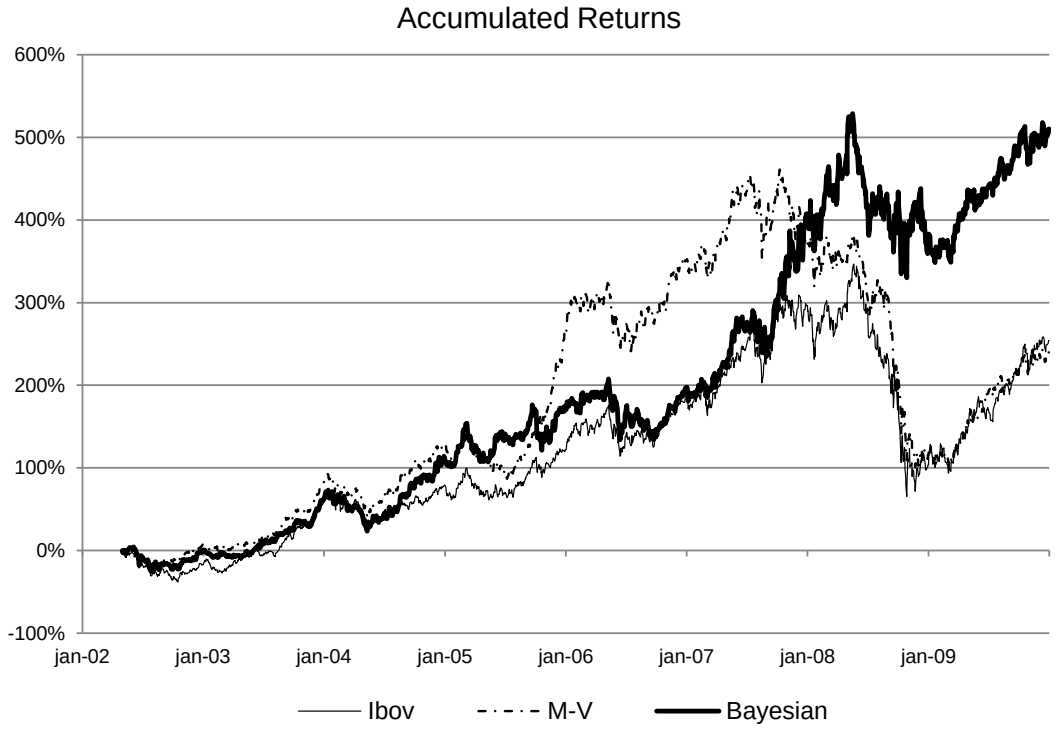

Figure 03 – Cumulative Returns of Models and Market

To analyze the reason for more stable returns of the BMM portfolios, two graphs were prepared (see Figures 04 and 05), showing the weights of the portfolios generated over the period of Figure 03. Figure 04 shows that the portfolios generated by the Classic Model MV repeatedly tend to invest in specific assets such as AMBEV-PN and PN-ULTRAPAR, those who had a direct impact by the economic crisis of 2008. In general only 18% of the assets that comprise the Bovespa index obtained positive returns between 2008 and 2009.

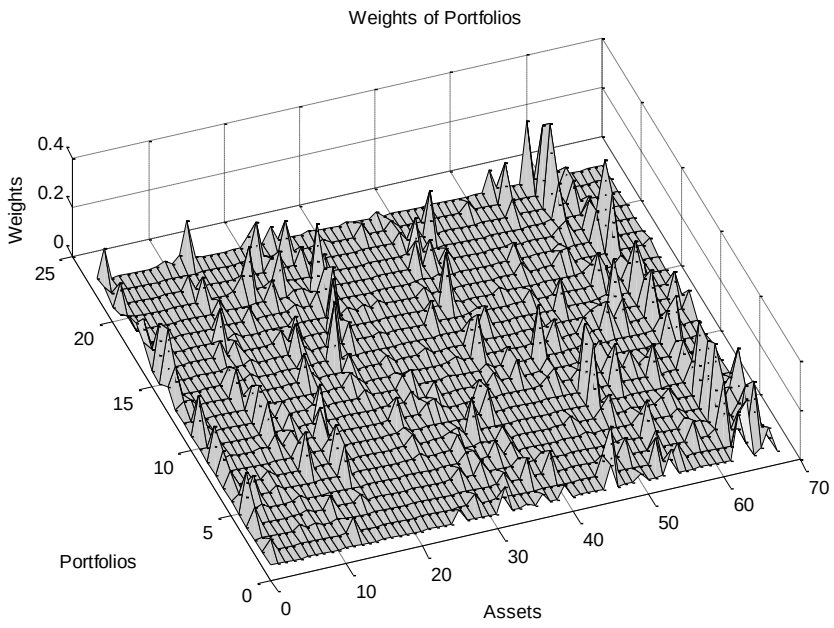

Figure 04 – Portfolio Weights of Mean Variance Model

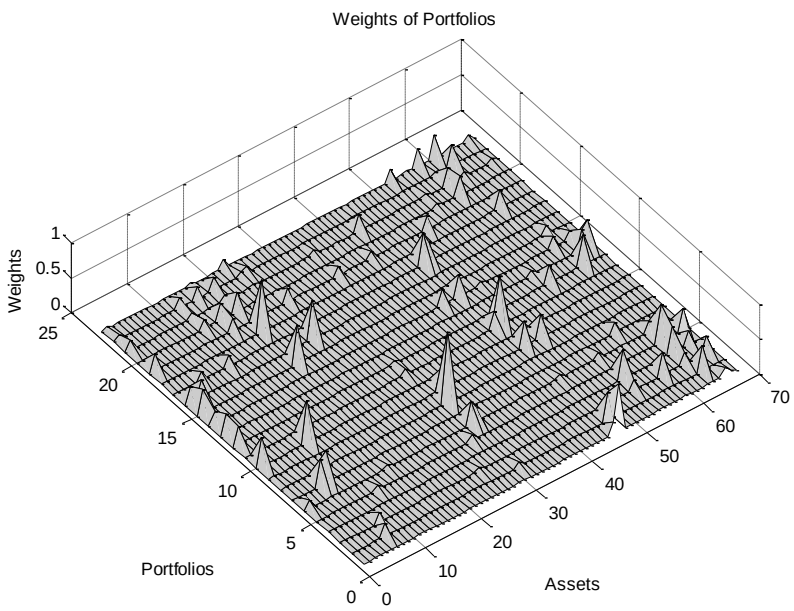

Figure 05 – Portfolio Weights of Bayesian Matrix Model

In contrast, the portfolios generated by BMM in Figure 05 show that they do not have a repeated emphasis on specific assets, although as mentioned above they have fewer assets in the composition of their portfolios. The advantage of reducing the number of assets is to incur fewer transaction costs, thus contributing to higher net returns.

Despite the final outcome of the model during the period 2002 to 2009 being higher than Ibovespa and the portfolios of the classical model, a robust analysis was made to compare the results between the benchmarks. In order to describe the outcome the model relative the variation of the investment time horizon, Table 02 was constructed from the results.

To construct Table 02, 1,960 portfolios were optimized for each time horizon, and evaluated by the cumulative effective return on investments in the period and the standard deviation of daily returns as a way of representing the risk. The difference between the sets of returns was evaluated by the Wilcoxon test, in accordance with the definition of the method. In Table 02, the investment horizons were measured from 30 to 180 days, with intervals of 30 days.

Table 02 – Statistical analysis of difference between the effective accumulated returns and standard deviation of portfolios built by each model and each time horizon of investment. (1) BMM – Bayesian Matrix Model, (2) MV – Mean Variance Model, and (3) Ibovespa - Bovespa Index.

Accumulated returns and risk over investment horizon

	Range of Investiment	Ibovespa	Sig.	BMM	Sig.	MV
Accumulated Returns	30 days	2.12%	-	2.80%	-	2.71%
	60 days	5.83%	-	5.95%	*	4.38%
	90 days	9.17%	-	9.44%	**	6.95%
	120 days	12.41%	*	12.16%	**	9.27%
	150 days	15.56%	-	15.57%	**	11.17%
	180 days	18.78%	-	18.68%	**	13.86%
Standard Deviation	30 days	1.22%	**	1.80%	**	1.55%
	60 days	1.82%	**	1.58%	**	1.26%
	90 days	1.82%	**	1.59%	**	1.28%
	120 days	1.82%	**	1.60%	**	1.30%
	150 days	1.82%	**	1.60%	**	1.30%
	180 days	1.82%	**	1.60%	**	1.32%

- \- No significance
- * Significant difference 5%
- ** Significant difference 1%

Also in Table 02, it is clear that the BMM in relation to the MV model shows significant differences in relation to risk and return. In other words, the new model tends to assume a little more risk in exchange for better returns. Regarding the Bovespa index, it was observed that the cumulative average returns of the portfolios of BMM showed no significant differences, except for returns at 30 and 120 days, which shows that the model tends to track the returns of market index.

The model analysis also shows that, despite the return track of the market index, the model risk is significantly lower than market risk. This conclusion is perceived in the period from 2002 to 2007 as shown in Figure 03.

Figure 06, built from Table 02, shows the average coefficients of the relationship between return and risk in relation to the investment time horizon. The graph shows that in longer maturities, the BMM model proposed, by taking the rate of Sharpe (1966) into consideration, tends to be higher than the benchmark, the market and the model of Markowitz (1952).

Using the Bayesian estimator based on the Bovespa index, the model presented near returns to the market, but with lower risk, which demonstrated that the model reduced the risk of the index because it was relocated. This shows that the model is suitable for building portfolios that aim to track the returns offered by the index, and assume less risk than the market.

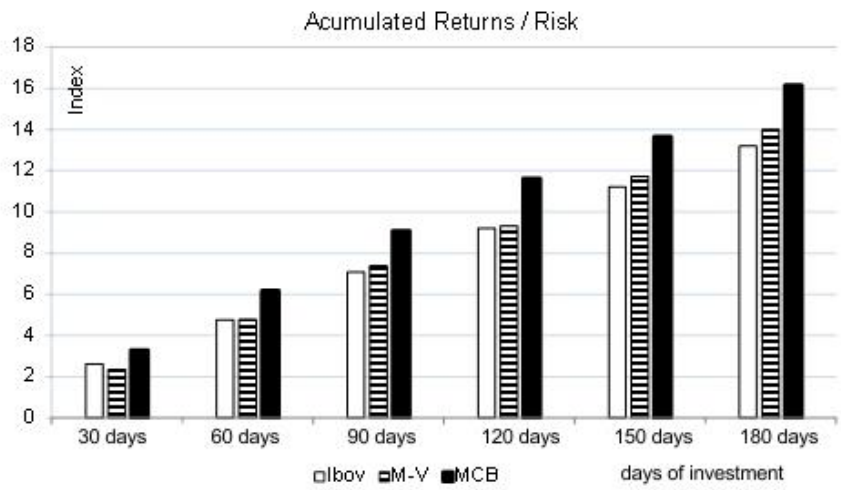

Figure 06 – Average of Sharpe Index of portfolios over investment horizon

In addition to the significant performance of the new Matrix Bayesian Model, it is possible that the matrix is improved. The model showed greater objectivity and performance of the benchmarks used. As suggested by Sharpe (1963), hybrid models tend to obtain more significant results than the classical MV. Therefore, new models are being developed as a way to achieve higher returns and lower risk.

4. Summary and conclusions

This study set out why and how a new model for portfolio selection was developed. Thus, a new model was developed to replace the covariance matrix, serving as a new input in the structure of Markowitz (1952). The model of the new matrix was built based on Bayesian analysis in line with Berger (1985) and a utility theory based on Friedman and Savage (1948). The new proposal is called BMM - Bayesian Matrix Model.

In the evaluation, the mean-variance model presented the problems highlighted by Michaud (1989) and Black and Litterman (1992), such as the deficiency in estimating parameters and emphasis on small-cap assets with the covariance matrix. The results showed that the model of Markowitz (1952) was not superior as to the returns relative to the market index, which according to Michaud (1989), requires the use of artificial constraints to minimize the estimation error and obtain better returns.

These problems of estimation of the covariance matrix have traditionally prompted new models to be developed such as the hybrid models by Sharpe (1963), Cohen and Pogue (1967) that promote disturbances in the traditional matrix. In an effort to corroborate these new solutions, this study drew up a new model for the matrix model used by Markowitz (1952).

According to a survey developed by Mendes and Leal (2005), the construction of good models depends on the use of a good estimator. Therefore, in accordance with the results of Polson and Tew (2000), the Bovespa index of the same set of assets was used as an estimator. The expectation was that the correlation between the series would contribute to a significant prediction of returns and the coefficient matrix of the objective function.

The new matrix had different characteristics from the traditional one in its structure, especially in the setting of the coefficients. Despite improving the optimization process, the matrix presented atypical points in more than one asset in contrast with traditional covariance, yet portfolios did not present an emphasis of assets with low capitalization (Michaud (1989)). As the estimator had lower sensitivity in the choice of market expectations, the returns absorbed less impact due to the depression in 2008. The results complement the results of Polson and Tew (2000).

The simulated returns during the period between May 2002 and December 2009 were superior to the classical model MV and the Bovespa Index, but taking risks that were a little more diversified than the model of Markowitz (1952). The robust analysis of the model, considering the time horizon, show returns that are near those of the Bovespa index, and are considered at lower risk than those in the market.

In general, the rate of Sharpe (1966) used as a comparison showed that the model becomes satisfactory, in return and risk, especially in longer maturities. These results corroborate the traditional concept that one can take a little more risk in exchange for higher returns, found by Boyle and Tian (2007) and refutes the idea that to get higher returns, it is not necessary to assume greater risk, as in Clarke et al (2006, 2011).

Another important result was to reduce the number of assets in the portfolios, there being an average of five, and the gain obtained in return and risk from the model contributes to the inquiry undertaken by Black and Litterman (1992),

about what diversification really is. The thought that an increase in the number of assets provides more diversification may be a misperception of the concept of diversification, which is characterized in a paradigm of building portfolios.

In relation to times of economic recession, the classic MV, as shown in Figure 03 and mentioned in Michaud (1989), showed a sensitivity to disturbances in the market. Therefore, according to Bauer (2004), the models need to consider atypical factors of the market. The results showed that the BMM model, by using a good estimator, reduced the sensitivity and/or delayed the impact that the market received in the subprime crisis. This can provide time for reaction and reallocation of investment. However, it is necessary to perform new tests to evaluate if the BMM considers atypical points in the market.

As a practical guideline, it is possible that this work may serve as a basis for creating an Exchange-Traded Fund ETF based on the model. This is an easy option and does not demand, from the investor, knowledge or time required for maintenance of a portfolio based on the BMM, with the guarantee of a minimum tracking of the performance of the market considering less risk.

Among the limitations of this study, it is important to highlight that the analysis was limited to the assets that comprise the Bovespa index. For other studies, the analysis of the model with other data sets is suggested as a way to re-evaluate the performance of the model. For future work, some suggestions are designed to prompt improvements in the development of the model and others that address the decision theory and Bayesian inference.

It is necessary to evaluate, in a comparative way, other options of estimators and models such as the Shrinkage Procedure. The use of dynamic optimization models is suggested. The scope of the theme enables new studies to be developed that will explore both the Brazilian and the international market. In addition to changing some of the model parameters, other artificial increments can be inserted to minimize the estimation errors.

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